Set-Valued Modeling for Drop-Point Constrained Dynamic Systems

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***Abstract*—This paper addresses the problem of drop-point constrained state modeling in the set-valued framework, where only drop-point information of state trajectories is available. Due to the lack of complete trajectory information, existing estimation methods with linear equality constraints struggle to achieve effective state prediction. This paper primarily investigates the integration of drop-point constraint information into the entire system state evolution process via convex optimization projection in the set-valued framework, aiming to reconstruct a linear dynamic system model. Subsequently, by employing multiple affine transformations of ellipsoids and designing a weight matrix, the smoothness of state trajectories is enhanced to better align with real motion patterns. Finally, through simulation experiments, we validate the significant advantages of the reconstructed system model over traditional unconstrained system models under the set-valued framework. Additionally, we demonstrate the impact of drop-point constraints on state trajectory evolution under different initial point conditions with the same drop point.**

***Index Terms*—Modeling, set-valued, drop-point constraints, dynamic systems**

I. Introduction

The study of dynamic system modeling issues in different motion scenarios has always been a focus of attention, see [1] and [2]. In practice, many dynamic systems operate under certain constraints, and these constraints encompass the inherent characteristics of the motion systems. Reasonable utilization of this prior information assists in refining system models (see [3]). Over the years, there has been exploration into innovative methods utilizing drop-point information to enhance tracking performance. This idea was initially proposed by [4], who introduced a modeling approach called forward-backward estimation fusion (FBEF) for ship monitoring. The key to this method lies in leveraging position measurements and potential drop points to improve tracking accuracy. Incorporating drop-point information into tracking strategies has gained good traction due to the premise that initial and final states exhibit stochastic behavior under specified boundary conditions, leading to significant developments. Building upon this foundation, researchers such as [5] and [6] have conducted systematic studies focusing on system modeling and state estimation methods for stochastic processes. Their efforts contribute to a deeper understanding of these processes and their applications in various domains. The latest advancements in this field have brought about a broader perspective, recognizing that, compared to reversible processes, conditional Markov (CM) processes offer a more natural generalization of traditional Markov processes. This realization has prompted comprehensive explorations into CM sequence modeling by [7] and [8]. Their research provides profound insights into general Gaussian CM sequences, covering both singular and nonsingular cases. Additionally, literature [9] and [10] investigate the construction and filtering algorithms of linear system models with drop-point constrained Gaussian noise assumptions.

In real-life scenarios, due to objective constraints, it's often impractical to obtain precise statistical properties of noise. This greatly increases the uncertainty when estimating system states using models. In many engineering applications, describing unknown noise with a bounded set is often more practical (see [11], [12]). In linear dynamic systems with unknown but bounded noise, set membership filters were initially proposed by [13]. Subsequently, these filters have garnered widespread attention and development in the study of dynamic systems under unknown but bounded noise settings (see [14]- [17]). These set membership filters offer a valuable framework for handling uncertainty and ensuring robustness under bounded noise. Recently, some scholars have introduced state constraint information in the set-valued framework, studying dynamic system models and filtering algorithms with state constraints, yielding promising results, as seen in [18] and [19]. These studies primarily focus on modeling and state estimation of constrained dynamical systems under unknown but bounded noise environments, enhancing the effectiveness and robustness of constrained dynamic system models in engineering applications.

In the stochastic framework, extensive research has been conducted on modeling dynamic systems with drop point constraints, yielding many mature results. However, modeling dynamic systems with drop point constraints in the set-valued framework remains an area that has not been widely explored. Therefore, this paper aims to conduct an in-depth analysis of linear dynamic systems with drop point constraints in the set-valued framework and establish appropriate robust models to improve the predictive performance of the system models. The main contributions of this paper are as follows: (i) Under the set-valued framework, utilizing convex optimization techniques to project the state of each time step onto the drop point constraint and naturally integrating drop point con-

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straint information into the state evolution process, achieving the reconstruction of the state model. (ii) Unlike traditional constrained state modeling methods (which typically assume that the process noise follows a known distribution, such as Gaussian distribution, uniform distribution, etc.), the method proposed in this paper sets the process noise as unknown but bounded noise, which is more natural and better suited to practical application requirements. (iii) Under the set-valued framework, we design an appropriate weight matrix and provide a specific expression to ensure that the trajectory of state evolution is smoother and more in line with actual motion laws.

II. Modeling of Linear State Model with Drop-Point Constraints

In the domain of target tracking, a fundamental aim is to predict the path of a target by precisely estimating its evolving state across time. The dynamic model governing target tracking typically describes the advancement of the target vector over time. This can be formulated for discrete-time linear dynamic systems as:

$$x_{k+1} = F_k x_k + w_k, 0 \leq k \leq N, \tag{1}$$

where k represents the time step, $x_k \in \mathbb{R}^n$ is the target motion state, $F_k \in \mathbb{R}^{n\times n}$ is an invertible state transfer matrix. The process noise $w_k \in \mathbb{R}^n$ is unknown but bounded. Assumed that w_k is contained within ellipsoid set $\mathcal{W}_k$, and satisfies:

$$\mathcal{W}_{w_k} = \{w_k \in \mathbb{R}^n | w_k^T P_k^{-1} w_k \leq 1\}, \tag{2}$$

where $P_k \in \mathbb{R}^{n\times n}$ represent the shape matrice of the ellipsoid set $\mathcal{W}_k$, and P_k is positive definite with appropriate dimensions.

This paper primarily considers state constraints at the final time step N, that is, the state x_N satisfies the following linear equality constraint

$$Cx_N = c. \tag{3}$$

Here, the matrix $C \in \mathbb{R}^{m\times n}$ (with $m < n$ and $\text{rank}(C) = m$) and the vector $c \in \mathbb{R}^m$ are known. It's essential to highlight that, despite the appearance of the drop point constraint (3) primarily affecting the final state x_N, its constraints reverberate across the entire evolution of the target state. This is due to the interdependence of state updates at each time step. For any time step s and k, where $0 \leq k \leq s \leq N$, the recursive relationship between x_k and x_s can be expressed using the state equation (1):

$$x_s = \Upsilon_{k,s} x_k + \eta_{k,s}, \tag{4}$$

where

$$\Upsilon_{k,s} = \prod_{i=1}^{s-k} F_{s-i}, \quad \Upsilon_{s,s} = I, \tag{5}$$

$$\eta_{k,s} = \begin{cases} \sum_{j=k}^{s-1} \Upsilon_{j+1,s} w_j, & k < s \\ 0, & k = s \end{cases}. \tag{6}$$

The identity matrix I has appropriate dimensions. As discussed previously in this paper, the information conveyed by the drop point constraint (3) exerts influence over the entire state evolution process, as illustrated in Fig. 1. Consequently, we now delve into strategies for effectively integrating this constraint information into the overall state updating process, aiming to improve the predictive performance of the state model.

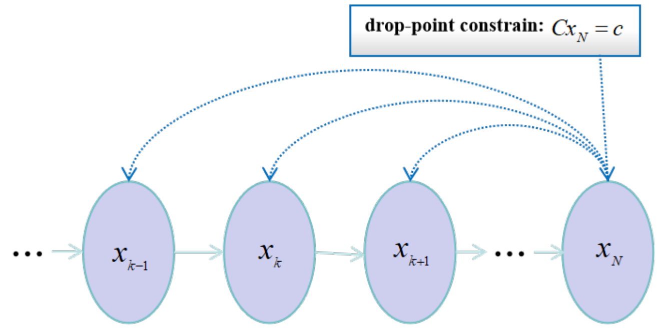

Fig. 1. State evolution with drop-point constraint.

Next, our attention turns to the pursuit of an optimal drop point state trajectory model from an optimization standpoint. Put differently, our objective is to identify a state trajectory subject to specific constraints that optimizes certain metrics, all while ensuring adherence to the drop point constraint (3). For ease of notation, we define the true constrained state as $\mathbf{x}_k = [x_k^T, x_N^T]^T$ and the relaxed constrained state as $\mathbf{x}_k^0 = [(x_k^0)^T, (x_N^0)^T]^T$. Let $\mathbf{C} = [\mathbf{0} \quad C]$, where the zero matrix $\mathbf{0}$ has appropriate dimensions. Hence, the true constraint (3) can be reformulated as

$$\mathbf{C}\mathbf{x}_k = Cx_N = c. \tag{7}$$

For any arbitrary vector $a \in \mathbb{R}^m$, the corresponding relaxed constraint is expressed as

$$\mathbf{C}\mathbf{x}_k^0 = Cx_N^0 = a. \tag{8}$$

The relationship between the true constrained state and the relaxed constrained state is

$$\begin{aligned} &\text{“True constrained state } \mathbf{x}_k\text{”} \\ &= \text{“Relaxed constrained state } \mathbf{x}_k^0|_{a=c}\text{”}. \end{aligned} \tag{9}$$

Due to the different choices of variable a, the relaxed state $\mathbf{x}_k^0$ is subject to distinct influences from drop point constraints, and its evolution trajectories are also different. Therefore, under the condition that all relaxation states $\mathbf{x}_k^0$ satisfy the true constraints, we hope to find an optimal trajectory that satisfies a certain distance and is as close as possible to the true constrained state trajectory $\mathbf{x}_k$. In this paper, the commonly used W-distance is considered, denoted as $\|x\|_W = \sqrt{x^T W^{-1} x}$, where W is a positive definite matrix. The problem described above can be transformed into solving the following optimization problem

$$\min_{\mathbf{x}_k^0} \ (\mathbf{x}_k^0 - \mathbf{x}_k) W_k{}^{-1} (\mathbf{x}_k^0 - \mathbf{x}_k)^T \tag{10}$$

$$s.t. \ \mathbf{C}\mathbf{x}_k^0 = a, \tag{11}$$

where $W_k = \begin{bmatrix} W_{k,1} & W_{k,2} \\ W_{k,2}^T & W_{k,3} \end{bmatrix} \in \mathbb{R}^{2n\times 2n}$ is a positive definite matrix, and $W_{k,\ell} \in \mathbb{R}^{n\times n}, \ell = 1,2,3$. Clearly, (10)-(11) form a quadratic programming (QP) problem that can be solved using the method of Lagrange multipliers. Thus, based on (10) and (11), we can establish the Lagrangian function for this optimization problem as

$$L(\mathbf{x}_k^0, \lambda) = \|\mathbf{x}_k^0 - \mathbf{x}_k\|_{W_k}^2 + \lambda^T(\mathbf{D}\mathbf{x}_k^0 - a), \tag{12}$$

where $\lambda \in \mathbb{R}^m$ denotes the Lagrange multiplier. Then, the first-order necessary conditions for the minimum of the optimization problem are

$$\frac{\partial L(\mathbf{x}_k^0, \lambda)}{\partial \mathbf{x}^0} = 2W_k(\mathbf{x}_k^0 - \mathbf{x}_k) + \mathbf{D}\lambda = \mathbf{0}, \tag{13}$$

$$\frac{\partial L(\mathbf{x}_k^0, \lambda)}{\partial \lambda} = \mathbf{D}\mathbf{x}_k^0 - a = \mathbf{0}. \tag{14}$$

From (13) and (14), we can derive the optimal solution of the relaxed constrained combined state $\mathbf{x}_k^0$ as

$$\mathbf{x}_k^0 = \mathbf{x}_k - W_k\mathbf{C}^T(\mathbf{C}W_k\mathbf{C}^T)^{-1}(\mathbf{C}\mathbf{x}_k - a). \tag{15}$$

Based on (7) and (9), it is easy to see that the right-hand side of equation (15) is equal to $\mathbf{x}_k$, i.e.,

$$\mathbf{x}_k - W_k\mathbf{C}^T(\mathbf{C}W_k\mathbf{C}^T)^{-1}(\mathbf{C}\mathbf{x}_k - c) = \mathbf{x}_k. \tag{16}$$

Therefore, by selecting an appropriate W_k, we can obtain the optimal relaxation state $\mathbf{x}_k^0$ that coincides with $\mathbf{x}_k$. On the other hand, considering $a = \mathbf{C}\mathbf{x}_k$, we can find that equation (16) implies that under the W-distance, $\mathbf{x}_k$ has the following decomposition form:

$$\mathbf{x}_k = B_k\mathbf{x}_k + (I - B_k)\mathbf{C}^\dagger c, \tag{17}$$

where $B_k = I - W_k\mathbf{C}^T(\mathbf{C}W_k\mathbf{C}^T)^{-1}\mathbf{C}$, and $\mathbf{C}^\dagger = \mathbf{C}^T(\mathbf{C}\mathbf{C}^T)^{-1}$. By substituting W_k into B_k, then

$$B_k = \begin{bmatrix} I & -W_{k,2}C^T(CW_{k,3}C^T)^{-1}C \\ \mathbf{0} & I - W_{k,3}C^T(CW_{k,3}C^T)^{-1}C \end{bmatrix}. \tag{18}$$

Observing the construction of $\mathbf{x}_k$ and substituting (18) into (17), we have

$$\begin{aligned} x_k = {} & x_k - W_{k,2}C^T(CW_{k,3}C^T)^{-1}Cx_N \\ & + W_{k,2}C^T(CW_{k,3}C^T)^{-1}c. \end{aligned} \tag{19}$$

If we let $s = N$ in (4) and substitute x_N into (19), then

$$\begin{aligned} x_k = {} & (I - W_{k,2}C^T(CW_{k,3}C^T)^{-1}C\Upsilon_{k,N})x_k \\ & + W_{k,2}C^T(CW_{k,3}C^T)^{-1}c \\ & - W_{k,2}C^T(CW_{k,3}C^T)^{-1}C\eta_{k,N}. \end{aligned} \tag{20}$$

Considering the substitution of state model (1) into (20) to establish a recursive form of state updates, it is important to note that the state evolution in (1) is unconstrained. Therefore, it is necessary to ensure that the state x_k at time N in (20) satisfies the drop point constraint (3), specifically by meeting $CW_{N,2}C^T = CW_{N,3}C^T$. In doing so, we can obtain the reconstructed drop-point constrained state model as follows:

$$\begin{aligned} x_k = {} & (I - \mathbf{B}_{k-1}\Upsilon_{k,N})F_{k-1}x_{k-1} + \mathbf{B}_{k-1}C^\dagger c \\ & + (I - \mathbf{B}_{k-1}\Upsilon_{k,N})w_{k-1} - \sum_{j=k}^{N-1}\mathbf{B}_{k-1}\Upsilon_{j+1,N}w_j, \end{aligned} \tag{21}$$

with $\mathbf{B}_{k-1} = W_{k,2}C^T(CW_{k,3}C^T)^{-1}C$, where $\Upsilon_{j+1,N}, j = k, \ldots, N-1$ is defined in (5).

Thus, by selecting appropriate $W_{k,2}$ and $W_{k,3}$, we can obtain a state model (21) that satisfies the drop point constraint (3), and this state model successfully incorporates the drop point constraint information into the entire process of state evolution. Additionally, we can achieve a uniform effect of the drop point constraint on the state evolution process by controlling the matrices $W_{k,2}$ and $W_{k,3}$. From equation (4), it can be seen that the state evolution process is mainly influenced by the transition matrix and process noise. Therefore, we hope to achieve our objectives by controlling the state cumulative transition matrix $\Upsilon_{j,N}$ (which contains the transition matrix information) and the process noise shape matrix P_j (which contains the process noise information), for $j = k, \ldots, N-1$. Next, we consider designing an adaptive weighted matrix W_k in the context of the drop point-constrained model, which is related to the state transition matrix and process noise. By doing so, we can obtain $W_{k,2}$ and $W_{k,3}$ to achieve the desired performance and stability of the model for the intended purpose.

III. Weight matrix selection for reconstruction model

In the context of a stochastic framework, such as when Gaussian white noise serves as the process noise, the selection of weights, denoted as W_k in (10), is a well-established practice. This choice, based on the covariance matrix of $\mathbf{x}_k^0$, serves to capture the relative uncertainty among its components. These weights enable the generation of a state trajectory that adheres to drop point constraints, ensuring a relatively uniform impact of such constraints on the state at each time step. However, in the set-valued framework, where the distribution of noise is unknown, direct calculation of the covariance matrix becomes impractical. Notably, there is a current gap in the literature regarding the specific determination of the weight matrix W_k in (10) when dealing with the relative uncertainty among components of an unknown but bounded variable $\mathbf{x}_k^0$. To address this challenge, our proposed approach involves enveloping the state $\mathbf{x}_k^0$ with a bounded ellipsoid in the set-valued framework. We quantify the correlation uncertainty between the components of $\mathbf{x}_k^0$ by leveraging its shape matrix. Similar to the principle of weight selection in the stochastic framework, we choose this shape matrix as the weight matrix W_k in (10).

$$\begin{aligned} & \mathbf{\Upsilon} = (\Upsilon_{k,N}, \cdots, \Upsilon_{N,N}), \mathbf{w} = (w_{k-1}^T, \cdots, w_{N-1}^T)^T \\ & \text{and} \quad \mathbf{D} = [I, \mathbf{0}, \cdots, \mathbf{0}]. \end{aligned} \tag{22}$$

Setting $s = N$ in (4), then we have

$$\mathbf{x}_k^0 = \begin{bmatrix} F_{k-1}x_{k-1}^0 + w_{k-1} \\ \Upsilon_{k-1,N}x_{k-1}^0 + \sum_{j=k-1}^{N-1} \Upsilon_{j+1,N}w_j \end{bmatrix}. \quad (23)$$

Let $k = 1$ in (22), then

$$\mathbf{w} = (w_0^T, \cdots, w_{N-1}^T)^T \triangleq \tilde{\mathbf{w}}. \quad (24)$$

Proposition 1. If $\tilde{\mathbf{w}}$ belongs to a known bounded ellipsoid $\mathcal{W}_{\tilde{\mathbf{w}}} := \mathcal{W}_{\tilde{\mathbf{w}}}(\mathbf{0}, P_{\tilde{\mathbf{w}}})$ with center at $\mathbf{0}$ and shape matrix

$$P_{\tilde{\mathbf{w}}} = \text{diag}(P_0, P_1, \cdots, P_{N-1}), \quad (25)$$

where $\text{diag}(\cdot)$ denotes the diagonal symbol of a matrix, and then we can derive the shape matrix $\mathbf{P}_{\Delta_k}$ of an ellipsoid covering the state $\mathbf{x}_k^0$, that is,

$$\mathbf{P}_{\Delta_k} = \begin{bmatrix} P_{k-1} & P_{k-1}\Upsilon_{k,N}^T \\ \Upsilon_{k,N}P_{k-1}^T & \sum_{i=k-1}^{N-1} \Upsilon_{i+1,N}P_i\Upsilon_{i+1,N}^T \end{bmatrix}. \quad (26)$$

Proof of Proposition 1. On the one hand, according to(23), we know that when the transition matrix is given, the uncertainty in updating from state $\mathbf{x}_{k-1}^0$ to $\mathbf{x}_k^0$ originates from the process noise. Thus, if we set

$$\check{\mathbf{x}}_k^0 = [(F_{k-1}x_{k-1}^0)^T, (\Upsilon_{k-1,N}x_{k-1}^0)^T]^T, \quad (27)$$

the uncertain portion (i.e., containing noise information portion) within $\mathbf{x}_k^0$ is represented by

$$\Delta_k = \mathbf{x}_k^0 - \check{\mathbf{x}}_k^0 = \mathbf{G}\mathbf{w}, \quad (28)$$

where $\mathbf{G} = [\mathbf{D}^T, \mathbf{\Upsilon}^T]^T$. In the set-valued framework, the distribution of variables is unknown, making it impossible to calculate their covariance matrix. Hence, we consider using an elliptical set

$$\mathcal{E}_{\Delta_k} = \{\Delta_k \in \mathbb{R}^n | (\Delta_k - \mathbf{c}_{\Delta_k})^T \mathbf{P}_{\Delta_k}^{-1} (\Delta_k - \mathbf{c}_{\Delta_k}) \leq 1\} \quad (29)$$

to cover the uncertainty Δ_k in this case, that is, $\Delta_k \in \mathcal{E}_{\Delta_k}$, where the shape matrix $\mathbf{P}_{\Delta_k}$ can represent the range of values for the uncertain term Δ_k.

On the other hand, we note that $\mathbf{w}$ in (22) consists of the last $N-k$ components of $\tilde{\mathbf{w}}$ intercepted. However the operation of intercepting can be realized by using a linear transformation of the matrix, i.e., $\mathbf{w} = H\tilde{\mathbf{w}}$. Here H is a matrix composed of elements 0 or 1 with the appropriate dimensions. Applying the assumptions in Proposition 1, we know that $\tilde{\mathbf{w}}$ belongs to the ellipsoid $\mathcal{W}_{\tilde{\mathbf{w}}}$, and due to the affine invariance of the ellipsoid [20], then we can obtain the affine ellipsoid $\mathcal{W}_{\mathbf{w}} = \mathcal{W}_{\mathbf{w}}(\mathbf{0}, P_{\mathbf{w}}) = H\mathcal{W}_{\tilde{\mathbf{w}}}$ of $\mathcal{W}_{\tilde{\mathbf{w}}}$ to cover $\mathbf{w}$. The shape matrix of ellipsoid $\mathcal{W}_{\mathbf{w}}$ is

$$P_{\mathbf{w}} = HP_{\tilde{\mathbf{w}}}H^T = \text{diag}(P_{k-1}, P_k, \cdots, P_{N-1}), \quad (30)$$

and the values of $P_{\mathbf{w}}$ are determined by the matrix $P_{\tilde{\mathbf{w}}}$ known from (25). Considering the notation in (28), we derive the Δ_k representation through a linear transformation of the vector $\mathbf{w}$. Leveraging the affine invariance of ellipsoids, a relationship emerges between the ellipsoid $\mathcal{E}_{\Delta_k}$ covering the vectors Δ_k and the ellipsoid $\mathcal{W}_{\mathbf{w}}$ covering the combined process noise vector $\mathbf{w}$:

$$\mathcal{E}_{\Delta_k} = \mathbf{G}\mathcal{W}_{\mathbf{w}}, \quad (31)$$

where the center and shape matrix associated with the ellipsoid satisfy the following conditions:

$$\mathbf{c}_{\Delta_k} = \mathbf{0}, \quad (32)$$

$$\mathbf{P}_{\Delta_k} = \mathbf{G}P_{\mathbf{w}}\mathbf{G}^T. \quad (33)$$

Taking into account (28), (32) and (33), it is easy to generate

$$\begin{aligned} \Delta_k \in \mathcal{E}_{\Delta_k} &= \{\Delta_k | (\Delta_k - \mathbf{c}_{\Delta_k})^T (\mathbf{P}_{\Delta_k})^{-1} (\Delta_k - \mathbf{c}_{\Delta_k}) \leq 1\} \\ &= \{(\mathbf{x}_k^0 - \check{\mathbf{x}}_k^0) | (\mathbf{x}_k^0 - \check{\mathbf{x}}_k^0)^T (\mathbf{G}P_{\mathbf{w}}\mathbf{G}^T)^{-1} (\mathbf{x}_k^0 - \check{\mathbf{x}}_k^0) \leq 1\}. \end{aligned} \quad (34)$$

If we make

$$\mathcal{E}_{\mathbf{x}_k^0} = \{\mathbf{x}_k^0 | (\mathbf{x}_k^0 - \check{\mathbf{x}}_k^0)^T (\mathbf{G}P_{\mathbf{w}}\mathbf{G}^T)^{-1} (\mathbf{x}_k^0 - \check{\mathbf{x}}_k^0) \leq 1\}, \quad (35)$$

then $\mathcal{E}_{\mathbf{x}_k^0}$ can be regarded as an ellipsoidal set covering state $\mathbf{x}_k^0$, with center $\check{\mathbf{x}}_k^0$ and shape matrix $\mathbf{G}P_{\mathbf{w}}\mathbf{G}^T = \mathbf{P}_{\Delta_k}$. This suggests that while ellipsoids $\mathcal{E}_{\Delta_k}$ and $\mathcal{E}_{\mathbf{x}_k^0}$ are centered differently, they possess identical shape matrices denoted by $\mathbf{P}_{\Delta_k}$, i.e.,

$$\mathbf{P}_{\Delta_k} = \mathbf{G}P_{\mathbf{w}}\mathbf{G}^T. \quad (36)$$

Substituting (22) and (30) into (36) yields

$$\begin{aligned} \mathbf{P}_{\Delta_k} &= \begin{bmatrix} P_{k-1} & \mathbf{D}P_{\mathbf{w}}\mathbf{\Upsilon}^T \\ \mathbf{\Upsilon}P_{\mathbf{w}}\mathbf{D}^T & \mathbf{\Upsilon}P_{\mathbf{w}}\mathbf{\Upsilon}^T \end{bmatrix} \\ &= \begin{bmatrix} P_{k-1} & P_{k-1}\Upsilon_{k,N}^T \\ \Upsilon_{k,N}P_{k-1}^T & \sum_{i=k-1}^{N-1} \Upsilon_{i+1,N}P_i\Upsilon_{i+1,N}^T \end{bmatrix}. \end{aligned} \quad (37)$$

This completes the proof of Proposition 1.

Remark. *Similar to the principle of selecting weighted least squares weight matrix in the random framework, based on* (35)*, we can choose the shape matrix $\mathbf{P}_{\Delta_k}$ of the ellipsoid $\mathcal{E}_{\mathbf{x}_k^0}$ covering the combined state $\mathbf{x}_k^0$ as the weight matrix W_k in the set-valued framework, which can achieve a similar weighting effect. Therefore, in practical applications, we recommend selecting*

$$W_k = \mathbf{P}_{\Delta_k} \quad (38)$$

as a superior choice for the weight matrix W_k in (10)*, which can result in smoother state trajectories. According to* (37) *and* (38)*, we acquire the expression of the weight matrix W_k in* (10)*. Consequently, in* (21)*, we can choose*

$$W_{k,2} = P_{k-1}\Upsilon_{k,N}^T, W_{k,3} = \sum_{i=k-1}^{N-1} \Upsilon_{i+1,N}P_i\Upsilon_{i+1,N}^T. \quad (39)$$

IV. Simulation and Discussion

The second and third sections of this paper respectively introduce the dynamic system model with drop point constraints induced by the linear dynamic system model in the set-valued framework, and the selection of the weight matrix. This section discusses the application of this model in trajectory modeling with drop point information. We consider a two-dimensional scenario, assuming that the state of the moving object at time step k is $x_k = (x, \dot{x}, y, \dot{y})$, where the position is (x, y) and the velocity is $(\dot{x}, \dot{y})$. The object takes off from coordinates (12 km, 0 km) with an initial velocity of (-240 m/s, 0 m/s) and lands at coordinates (0 km, 12 km). Assume that the angle between the object's heading and the horizontal direction (measured clockwise from due east) at the time it reaches the drop position is recorded as α, and that the directions due east and due north are positive directions in two-dimensional coordinates. To establish a trajectory model between the start point and the drop point, we utilize a relaxed state model induced by an unconstrained (almost constant velocity) model (1), namely:

$$x_{k+1}^0 = F_k x_k^0 + w_k, w_k \in \mathcal{W}_{w_k} := \mathcal{W}_{w_k}(0, P_k), \tag{40}$$

with the following parameters for $k \in [1, N]$

$$F_k = \text{diag}(F, F), \tag{41}$$

$$P_k = g^2 \text{diag}(P, P), \tag{42}$$

where $F = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}, P = q \begin{bmatrix} T^3/3 & T^2/2 \\ T^2/2 & T \end{bmatrix}$, $T = 1$ s, $g = 9.8$ m/s^2 and $N = 80$ s. T is the sampling interval (i.e., interval between $k-1$ and k).

By constructing W_k as described in (38), our objective is to enhance the realism of the state evolution pattern through the manipulation of the state transition matrix and integration of process noise. Let $W_{k+1,2} = P_{k+1}\Upsilon_{k+1,N}^T$ and $W_{k+1,3} = \sum_{j=k+1}^{N} \Upsilon_{j,N} P_{k+1} \Upsilon_{j,N}^T$, (21) can be rewritten as

$$\begin{aligned} x_{k+1} = {} & (I - \mathbf{B}_k \Upsilon_{k+1,N}) F_k x_k + \mathbf{B}_k C^{\dagger} c \\ & + (I - \mathbf{B}_k \Upsilon_{k+1,N}) w_k - \sum_{j=k+1}^{N-1} \mathbf{B}_k \Upsilon_{j+1,N} w_j, \end{aligned} \tag{43}$$

where $\mathbf{B}_k = W_{k+1,2} C^T (C W_{k+1,3} C^T)^{-1} C$. The parameters in the state drop point constraint (3) are set with the following values:

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\cot(\alpha) \\ 0 & 0 & 1 & 0 \end{bmatrix} \text{ and } c = \begin{bmatrix} 0 \\ 0 \\ 12000 \end{bmatrix}.$$

By implementing the specified parameter configurations into the state model, one with drop point constraint (43) and the other a relaxed version (41), we conducted a random generation of 20 state trajectories for each model. These trajectories are depicted in Fig. 2 (with $\alpha = 90°$ at the drop point) and Fig. 3 (with $\alpha = 0°$ at the drop point), illustrating distinct evolution patterns. In these figures, trajectories derived from the relaxed state model (x_k^0) exhibit scattered and irregular evolution patterns, failing to pass the predetermined drop point. Conversely, trajectories generated by the drop point-constrained state model (x_k) demonstrate a coherent trend, consistently converging towards the drop point. Importantly, all trajectories from the drop-point constrained model are guaranteed to pass through the specified drop point, validating successful integration of drop point constraints. Moreover, all generated trajectories maintain smooth and natural transitions, indicating the appropriateness of the chosen weight matrix W_k, which uniformly imposes drop point constraints across each moment of state evolution, thereby enhancing predictive accuracy.

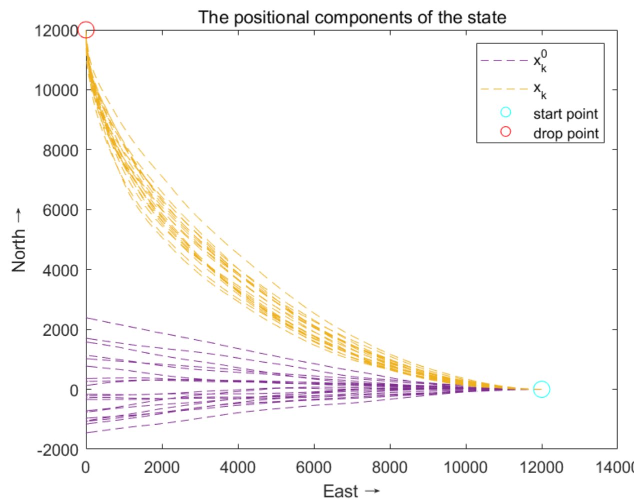

Fig. 2. The trajectories of x_k^0 and x_k, with $\alpha = 90°$ at the drop point

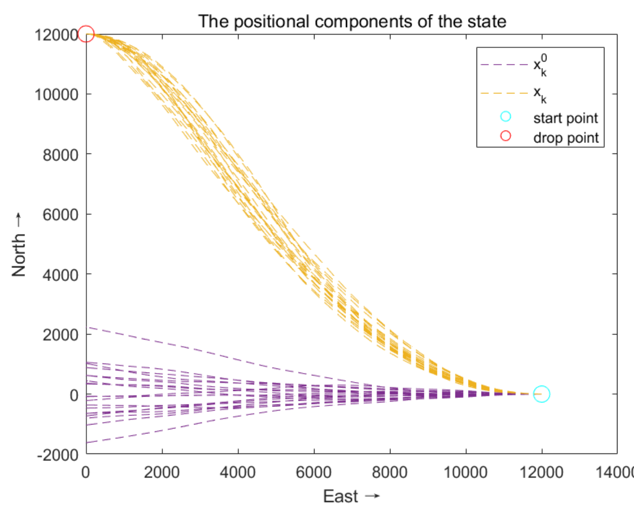

Fig. 3. The trajectories of x_k^0 and x_k, with $\alpha = 0°$ at the drop point

When arriving at the drop point with heading angles $\alpha = 90°$ and $\alpha = 0°$ relative to the horizontal direction, we randomly selected 20 start points and set the same drop point location, plotting the trajectory patterns of two groups (each consisting of 20 curves) of x_k^0 as shown in Figure 4 and Figure 5, respectively. From these two figures, we can observe different evolution trends in the trajectories of the two groups due to the different heading angles α at the drop point. Additionally, despite selecting 20 different random start points, since they all share a common drop point, the trajectories in both groups consistently progress towards the drop point and eventually reach the designated location. This demonstrates the influence of drop point constraints on the evolution process of state trajectories.

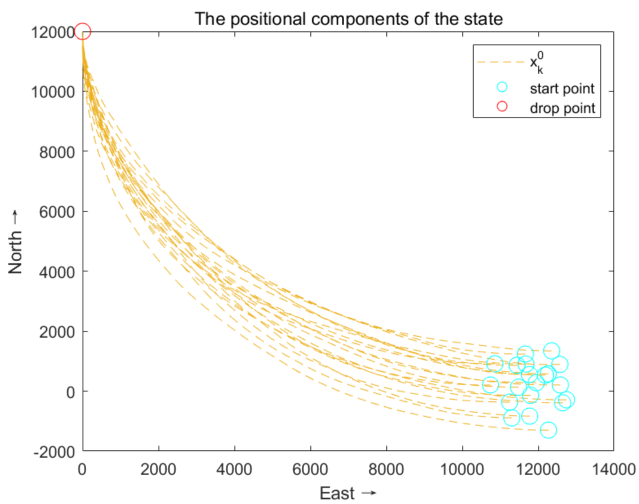

Fig. 4. Trajectory evolution of all x_k^0 for different start points and the same drop point condition, $\alpha = 90°$ at the drop point

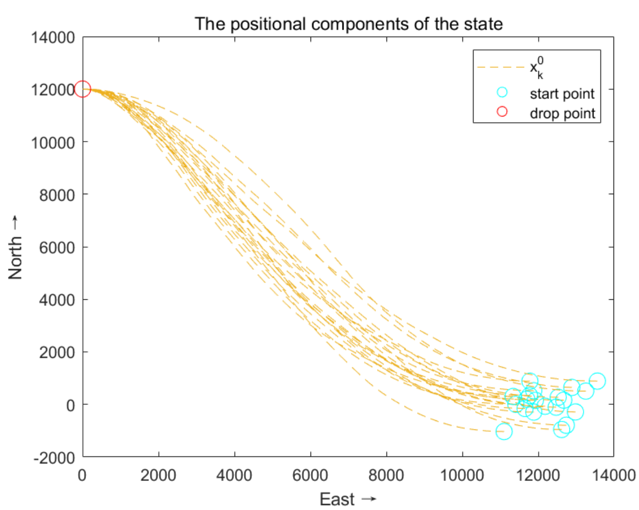

Fig. 5. Trajectory evolution of all x_k^0 for different start points and the same drop point condition, $\alpha = 90°$ at the drop point

V. Conclusion

This paper delves into the modeling of dynamic systems with drop-point constraints in the set-valued framework, yielding valuable insights. We quantify drop-point constraints as known components of state evolution and integrate constraint information into the sequence of state updates through state projection, establishing a novel system model. Building upon this, we devise appropriate weight matrice to ensure smooth and natural evolution trends of state trajectories. Simulation experiments demonstrate superior performance of our proposed system model compared to traditional relaxed state models, highlighting the profound impact of drop-point constraints on the evolution of state trajectories under different initial points and identical drop points settings. In summary, our work organically integrates drop point constraint into dynamic systems in the set-valued framework, achieving model reconstruction and offering new perspectives to the field of system modeling.

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